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Journal of Fluids and Structures 19 (2004) 985–1003

JOURNAL OF FLUIDS AND STRUCTURES

www.elsevier.nl/locate/jnlabr/yjfls

Response of half–full horizontal cylinders under transverse excitation

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Received 13 June 2003; accepted 28 April 2004

Available online 10 August 2004

Abstract

The present work investigates the response of half–full horizontal cylindrical vessels under external excitation in the transverse direction. A two dimensional mathematical model is developed to describe sloshing effects in rigid vessels. The velocity potential is expressed in a series form, where each term is the product of a time function and the associated spatial function. In this geometrical configuration the spatial functions are not orthogonal and the problem is not separable. Application of the boundary conditions results in a system of ordinary linear differential equations, which are solved numerically. Sloshing frequencies of half–full horizontal cylinders are computed, and hydrodynamic forces are calculated. Under harmonic excitation, the formulation results in a system of linear equations, allowing for a semi-analytical solution. A simplified version of the mathematical model is also developed, which considers the first two terms of the series and results in an elegant solution. Furthermore, assuming a beam-type deformation of the container, the simplified formulation can be extended to approximate the coupled response of the container-liquid system. Using this formulation, the response of a typical pressure vessel under ground-motion excitation is calculated and the effects of wall deformation are demonstrated.

Keywords: Sloshing; Pressure vessel; Fluid–structure interaction; Seismic response; Semi-analytical solution

1. Introduction

The presence of a free surface in partially filled liquid containers allows for fluid motions relative to the container. This phenomenon, referred to as "liquid sloshing", is generally caused by external tank excitation (e.g. by an earthquake), and may have a significant influence on the response of the container.

Sloshing has been often considered as a typical linear eigenvalue problem, which represents the oscillations of the free surface of an ideal liquid inside a stationary container. The solution provides the natural frequencies of fluid oscillation (sloshing frequencies) and the corresponding sloshing modes, and depends strongly on the shape of the container. In the case of externally excited container, sloshing becomes a transient problem, and its solution provides the fluid motion with respect to the container, as well as the time history of hydrodynamic pressures and forces on the container wall. In both problems, assuming ideal fluid, the fluid flow is described through a velocity potential function satisfying the Laplace equation within the fluid, the kinematic condition on the tank wall, and the kinematic and dynamic free-surface conditions.

For nondeformable rectangular and vertical cylindrical containers, the sloshing problem can be solved analytically, using separation of variables, and the corresponding sloshing modes are mutually orthogonal and uncoupled. For other

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0889-9746/$ - see front matter
doi:10.1016/j.jfluidstructs.2004.04.014

geometries (e.g. horizontal cylinders or spheres) exact analytical solutions may not be available, and the use of numerical methods becomes necessary. Budiansky (1960) has examined sloshing effects in nondeformable circular canals, and reported numerical values of modal frequencies and hydrodynamic forces. Moiseev and Petrov (1966) described the application of Ritz variational method for the numerical calculation of sloshing frequencies in vessels of various geometries, including the case of a horizontal cylindrical container. Fox and Kuttler (1981, 1983) obtained upper and lower bounds for the values of sloshing frequencies in a semi-circular canal using conformal mapping and the method of intermediate problems. McIver (1989) considered horizontal cylindrical containers, filled up to an arbitrary height, reformulating the eigenvalue-sloshing problem in terms of integral equations, which were solved numerically. More recently, McIver and McIver (1993) presented analytical methods to obtain upper and lower bounds of sloshing frequencies in horizontal cylinders.

Generally, the analysis of sloshing in horizontal cylindrical and spherical vessels filled up to an arbitrary height requires a numerical solution. However, for the particular case of a half–full horizontal cylinder and sphere it is possible to develop an analytical solution. Evans and Linton (1993) presented a series-type (semi-analytical) solution of the eigenvalue-sloshing problem in half–full horizontal cylindrical containers and half–full spherical vessels, expanding the velocity potential in terms of nonorthogonal bounded harmonic spatial functions. In a recent publication (Papaspyrou et al., 2003), the solution of Evans and Linton (1993) was extended to calculate sloshing effects in externally excited half–full spheres.

The present work is aimed primarily at calculating sloshing effects in half–full horizontal cylindrical containers under external excitation in the transverse direction, extending the analytical formulation of Evans and Linton (1993). In particular, the main objective of the paper is the solution of externally induced liquid sloshing in half–full cylinders under transverse excitation, through a semi-analytical manner, without implementing finite difference or finite element approximations. Expanding the velocity potential in bounded series in terms of arbitrary time functions and their associated nonorthogonal spatial functions, a system of ordinary linear differential equations is obtained and, subsequently, hydrodynamic pressures and forces are computed for arbitrary external excitation. The particular case of harmonic excitation is examined. Dissipation effects are also taken into account in the form of a Rayleigh damping matrix. In addition, a simplified methodology is developed considering only the first two terms of the series, which yields a linear oscillator equation and gives rise to an equivalent mechanical model.

Furthermore, the present study is aimed at estimating the effects of container–wall deformation on the overall response. Wall deformation effects have been studied extensively in the case of vertical cylinders. Those works have assumed that wall deformation affects only the "impulsive" part of the motion describing the motion of the container through either simple assumed-shape functions (Veletsos and Yang, 1977; Fischer, 1979), or more elaborate shell models (Chu, 1963; Haroun and Housner, 1981; Haroun, 1983; Natsiavas, 1988; Rammerstorfer et al., 1990; Gupta, 1995). On the other hand, the effects of wall deformation on the response of horizontal cylinders have not been investigated. In industrial applications, those vessels are rather thick to resist high internal pressure and, therefore, shell-type vibration modes may not be significant. However, relatively long horizontal cylindrical vessels ($L/R \geqslant 10$), quite common in petrochemical industries and refineries, exhibit a beam-type deformation, which may affect the overall response under transverse excitation. Using the aforementioned simplified sloshing formulation and considering an assumed-shape beam-type approach for cylinder deformation, it is possible to estimate the response of the coupled container–liquid system for half–full deformable containers under transverse excitation.

2. Sloshing under transverse excitation

The fluid is contained in a half–full horizontal cylindrical vessel of radius R, with the y-axis of the coordinate system xyz pointing vertically downwards (Fig. 1), and the geometry is described in terms of the cylindrical coordinates r, θ, z. The container undergoes an arbitrary motion in the direction of the x-axis with displacement $X(t)$. For the purposes of the present sloshing formulation, the vessel is assumed rigid (nondeformable). Wall deformation effects are considered in the next section.

2.1. Problem formulation

Assuming inviscid fluid, the flow is described by a velocity potential function $\Phi(r, \theta, z, t)$, which satisfies Laplace equation within the fluid volume:

$$\nabla^2 \Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0, \quad r < R, -\pi/2 < \theta < \pi/2, \ 0 < z < L. \tag{1}$$

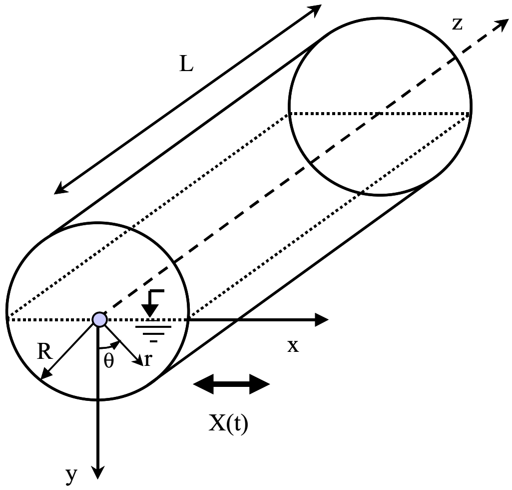

Fig. 1. Configuration of half–full horizontal cylinder; excitation $X(t)$ is considered in the transverse direction (axis x).

The free-surface elevation is assumed to be sufficiently small to allow linearization of the problem, so that Φ is subjected to the linearized dynamic and kinematic free-surface conditions

$$\frac{\partial \Phi}{\partial t} - g\eta = 0 \quad \text{at } \theta = \pm\pi/2,\ r<R,\ 0<z<L \tag{2}$$

and

$$\pm\frac{1}{r}\frac{\partial \Phi}{\partial \theta} + \frac{\partial \eta}{\partial t} = 0 \quad \text{at } \theta = \pm\pi/2,\ r<R,\ 0<z<L, \tag{3}$$

respectively, where g is the gravitational constant and $\eta = \eta\ (r,z,t)$ is the free-surface elevation. Combination of Eqs. (2) and (3) leads to the following mixed boundary condition:

$$\frac{\partial^2 \Phi}{\partial t^2} \pm \frac{g}{r}\frac{\partial \Phi}{\partial \theta} = 0 \quad \text{at } \theta = \pm\pi/2,\ r<R,\ 0<z<L. \tag{4}$$

Assuming external excitation along the (transverse) x-axis, the sloshing potential should satisfy the kinematic condition at the container walls

$$\frac{\partial \Phi}{\partial r} = \dot{X}(t)\sin\theta \quad \text{at } r = R, -\pi/2<\theta<\pi/2,\ 0<z<L \tag{5}$$

and

$$\frac{\partial \Phi}{\partial z} = 0 \quad \text{at } z = 0, L, -\pi/2<\theta<\pi/2,\ 0<r<R. \tag{6}$$

Subsequently, Φ is decomposed in two parts:

$$\Phi(r,\theta,z,t) = \varphi_U(r,\theta,z,t) + \tilde{\varphi}(r,\theta,z,t), \tag{7}$$

where $\varphi_U(r,\theta,z,t)$ and $\tilde{\varphi}(r,\theta,z,t)$ are the "uniform motion" velocity potential and the potential related to sloshing, respectively. The velocity potential φ_U corresponds to a rigid-body motion of the fluid, which follows exactly the motion of the external excitation source, and the velocity potential $\tilde{\varphi}$ represents the relative motion of fluid within the container due to sloshing. The uniform motion potential φ_U is taken as

$$\varphi_U = \dot{X}(t)x = \dot{X}(t)r\sin\theta, \tag{8}$$

which satisfies the Laplace equation (1), and the kinematic condition (5) at the container wall $r = R$. Thus, the unknown potential $\tilde{\varphi}$ associated with sloshing, should satisfy the Laplace equation within the fluid region and the following boundary conditions:

$$\frac{\partial^2 \tilde{\varphi}}{\partial t^2} \pm \frac{g}{r}\frac{\partial \tilde{\varphi}}{\partial \theta} = -\frac{\partial^2 \varphi_U}{\partial t^2} \quad \text{at } \theta = \pm\pi/2,\ r<R,\ 0<z<L \tag{9}$$

$$\frac{\partial\tilde{\varphi}}{\partial r} = 0 \quad \text{at } r = R, -\pi/2 < \theta < \pi/2,\ 0 < z < L \tag{10}$$

and

$$\frac{\partial\tilde{\varphi}}{\partial z} = 0 \quad \text{at } z = 0, L, -\pi/2 < \theta < \pi/2,\ 0 < r < R. \tag{11}$$

Then the velocity potential $\tilde{\varphi}$ is written as

$$\tilde{\varphi}(r,\theta,z,t) = \sum_{p=0}^{\infty} \tilde{\varphi}_p(r,\theta,t)\cos\left(\frac{p\pi z}{L}\right), \qquad r < R, -\pi/2 < \theta < \pi/2, 0 < z < L, \tag{12}$$

thereby ensuring that boundary condition (11) is satisfied provided $p = 0,1,2,3,\ldots$. Considering the boundary condition (10), it can readily be shown that, due to the type of external excitation, only the term corresponding to the mode $p = 0$ is nonzero (i.e. the term which is constant with respect to z), while all other terms ($p = 1,2,3,\ldots$) vanish. Thus, the initial problem reduces to a strictly two dimensional problem consisting of calculating a velocity potential $\varphi = \tilde{\varphi}_0(r,\theta,t)$, which satisfies the Laplace equation in a lower half-disk, the mixed boundary condition at the free surface,

$$\frac{\partial^2\varphi}{\partial t^2} \pm \frac{g}{r}\frac{\partial\varphi}{\partial\theta} = -\frac{\partial^2\varphi_U}{\partial t^2}, \qquad \theta = \pm\pi/2,\ r < R, \tag{13}$$

and the kinetic boundary condition at the container wall,

$$\frac{\partial\varphi}{\partial r} = 0, \quad \text{at } r = R, -\pi/2 < \theta < \pi/2. \tag{14}$$

2.2. Sloshing solution

A solution for the unknown function φ is considered in a series form as

$$\varphi = \sum_{n=1}^{\infty} \dot{q}_n(t)\varphi_n(r,\theta) = \sum_{n=1}^{\infty} \dot{q}_n(t) r^n \sin(n\theta), \qquad r < R, -\pi/2 < \theta < \pi/2, \tag{15}$$

where $q_n(t)$ are unknown arbitrary time functions, and $\varphi_n(r,\theta) = r^n \sin(n\theta)$ are the corresponding spatial functions. For the purposes of the present study, motivated by the methodology of Evans and Linton (1993), the expression for the unknown potential is rewritten in the following form:

$$\varphi(r,\theta,t) = \sum_{n=1}^{\infty} [\dot{q}_{2n-1}(t) r^{2n-1}\sin(2n-1)\theta + \dot{q}_{2n}(t) r^{2n}\sin(2n\theta)], \tag{16}$$

separating odd and even terms of the series. Substituting Eqs. (16) and (8) into Eq. (13) and equating terms of equal power in r the following relations are obtained:

$$q_2(t) = \frac{1}{2g}\ddot{q}_1(t) + \frac{1}{2g}\ddot{X}(t) \tag{17}$$

and

$$q_{2n}(t) = \frac{1}{2ng}\ddot{q}_{2n-1}(t), \qquad \text{for } n > 1. \tag{18}$$

Eqs. (17) and (18) are substituted back into Eq. (16) and then applying the boundary condition at the container wall, expressed by Eq. (14), the following equation is obtained:

$$\sum_{n=1}^{\infty}\left\{\frac{R^{2n-1}}{g}\sin(2n\theta)\ddot{q}_{2n-1}(t) + (2n-1)R^{2n-2}\sin(2n-1)\theta q_{2n-1}\right\} = -\frac{R}{g}\sin(2\theta)\ddot{X}(t). \tag{19}$$

Subsequently, applying the integral operator

$$I_s = \int_0^{\pi/2} \ldots \sin(2s-1)\theta\, \mathrm{d}\theta, \qquad s = 1,2,3\ldots, \tag{20}$$

on Eq. (19) and conducting some mathematical manipulations, the following infinite system of second-order ordinary linear differential equation is obtained:

$$[\mathbf{M}]\{\ddot{q}\} + [\mathbf{K}]\{q\} = -\{\gamma\}\ddot{X}. \tag{21}$$

In Eq. (21), [**M**] is a nonsymmetric square matrix, [**K**] is a diagonal matrix, and $\{\gamma\}$ is a vector, where

$$M_{sn} = \frac{2n(-1)^{n+s}}{n^2 - (s-1/2)^2} R^{2n-1}, \quad n = 1,2,3\ldots \quad \text{and} \quad s = 1,2,3\ldots, \tag{22}$$

$$K_{nn} = (2n-1)\pi g R^{2n-2}, \quad n = 1,2,3\ldots, \tag{23}$$

$$\gamma_s = \frac{8(-1)^{s+1}}{3+4s-4s^2} R, \quad s = 1,2,3\ldots, \tag{24}$$

and $\{q\}$ is the unknown vector with components $q_{2n-1}(t), n = 1,2,\ldots$.

The system of Eqs. (21) expresses the dynamic equilibrium of the system, where $\{q\}$ is the vector of unknown generalized coordinates, [**M**] and [**K**] may be considered as the mass and stiffness matrices of the system, respectively, and $\{\gamma\}$ is the vector expressing the contribution (participation) of external excitation on the dynamic equilibrium.

Upon numerical solution of the truncated system of Eqs. (21) in terms of $q_{2n-1}(t)$, functions $q_{2n}(t)$ should be determined, so that the potential φ associated with sloshing is completely defined. To calculate functions $q_{2n}(t)$, it is straightforward to use Eqs. (17) and (18), which express $q_{2n}(t)$ in terms of $\ddot{q}_{2n-1}(t)$ and the acceleration of the external excitation $\ddot{X}(t)$. Implications may arise when the second derivatives of $q_{2n}(t)$ are calculated, to compute hydrodynamic pressure and forces (as described in Section 2.3). This requires calculation of the fourth derivatives of $q_{2n-1}(t)$ and $X(t)$. In the case of an irregular function $\ddot{X}(t)$ (e.g. a seismic ground motion), $\ddot{q}_{2n-1}(t)$ are also irregular functions, containing very sharp variations within very small time intervals, and their numerical differentiation may lead to erroneous results. It is possible to avoid such a numerical difficulty, under the observation that vector $\{\gamma\}$ consists of the same elements with the first column of matrix [**M**]. Therefore, Eqs. (21) can be written as follows:

$$[\mathbf{M}]\{\ddot{Q}\} + [\mathbf{K}]\{q\} = \{0\}, \tag{25}$$

where

$$\{\ddot{Q}\} = \begin{bmatrix} \ddot{q}_1 + \ddot{X} \\ \ddot{q}_3 \\ \ddot{q}_5 \\ \vdots \end{bmatrix}. \tag{26}$$

On the other hand, Eqs. (17) and (18) can be written as

$$\{\bar{q}\} = \frac{1}{2ng}\{\ddot{Q}\}, \tag{27}$$

where $\{\bar{q}\}$ is a vector with components $q_{2n}(t)$, n=1,2,3… . Combining Eqs. (25) and (27), vector $\{\bar{q}\}$ is calculated as follows:

$$\{\bar{q}\} = -\frac{1}{2ng}[\mathbf{M}]^{-1}[\mathbf{K}]\{q\}. \tag{28}$$

Thus, functions $q_{2n}(t)$ are calculated algebraically from functions $q_{2n-1}(t)$, and the double differentiation of $q_{2n}(t)$ becomes a trivial procedure, since the second derivatives of $q_{2n-1}(t)$ are obtained from the solution of Eqs. (21).

To account for dissipation effects, a damping term proportional to the first derivative of the generalized coordinates is introduced in Eqs. (21), so that

$$[\mathbf{M}]\{\ddot{q}\} + [\mathbf{C}]\{\dot{q}\} + [\mathbf{K}]\{q\} = -\{\gamma\}\ddot{X}. \tag{29}$$

In the above equations, [**C**] is a square matrix, herein considered in the form of a Rayleigh damping matrix

$$[\mathbf{C}] = \partial_0[\mathbf{M}] + \partial_1[\mathbf{K}], \tag{30}$$

where ∂_0 and ∂_1 are constants.

It is interesting to note that in previous works, a different approach was proposed to model damped free-surface systems [Faltinsen (1978); Isaacson and Subbiach (1991)], which enables the use of potential theory and introduces a damping term proportional to the potential related to sloshing in the dynamic boundary condition at the free surface:

$$\frac{\partial \Phi}{\partial t} + v\tilde{\varphi} - g\eta = 0 \quad \text{at } \theta = \pm\pi/2, \ r<R, \ 0<z<L. \tag{31}$$

The second term on the left-hand side represents a force, which opposes particle velocity, and the proportionality constant v is a viscosity coefficient. If boundary condition (31), instead of (2), is implemented in the present formulation, it results in a system of ordinary differential equations identical to that of Eq. (29), with $[\mathbf{C}] = v[\mathbf{M}]$, a special form of Rayleigh damping.

2.3. Hydrodynamic pressures and forces

Once the velocity potential φ associated with sloshing is calculated, the hydrodynamic pressure at any location can be computed from the linearized Bernoulli equation and the total horizontal force acting on the container is obtained by an appropriate integration of the pressure on the hemispherical wall. The total hydrodynamic force f_T on the container wall is the sum of the ''uniform motion'' force f_U, the force associated with sloshing f_S and the container's inertia force f_C given by

$$f_U = -\rho \int_A \frac{\partial \varphi_U}{\partial t} (\mathbf{e_x} \cdot \mathbf{n})\, \mathrm{d}A, \tag{32}$$

$$f_S = -\rho \int_A \frac{\partial \varphi}{\partial t} (\mathbf{e_x} \cdot \mathbf{n})\, \mathrm{d}A \tag{33}$$

and

$$f_C = -m_C \ddot{X}(t), \tag{34}$$

respectively, where ρ is the fluid mass density, A is the wet surface of the cylinder, and m_C is the mass per unit length of the cylindrical container. Using Eqs. (8) and (16), f_U and f_S are calculated as follows:

$$f_U = -\rho \ddot{X}(t) R^2 \int_0^L \int_{-\pi/2}^{\pi/2} \sin^2\theta \, \mathrm{d}\theta \, \mathrm{d}z = -\left(\rho \frac{\pi R^2}{2}\right) \ddot{X}(t) = -m_L \ddot{X}(t), \tag{35}$$

$$f_S = -\rho R^2 \sum_{n=1}^{\infty} R^{2n-2} [\ddot{q}_{2n-1}(t) Y_{2n-1} + R \ddot{q}_{2n}(t) Y_{2n}], \tag{36}$$

where $m_L = \rho\pi R^2/2$ is the liquid mass per unit length of the half–full horizontal cylinder container, and

$$Y_k = \int_{-\pi/2}^{\pi/2} \sin(2k\theta)\sin\theta \, \mathrm{d}\theta = (-1)^{k+1} \frac{4k}{4k^2 - 1}, \quad k = 1, 2, 3\ldots . \tag{37}$$

Since the pressure is always normal to the wall of the container, the total hydrodynamic force direction always passes through the center of the cross-section of horizontal cylinder.

3. Simplified analysis and fluid–vessel interaction

The problem formulation of a half–full container response is significantly simplified if only the first two terms of the series solution are considered [Eq. (16) with $n = 1$]. In such a case, an approximate solution is obtained and sloshing is described in terms of a linear oscillator, which gives rise to an equivalent mechanical model. Using this solution and an assumed-shape for the deformation of the vessel, the coupled response of the interacting liquid–vessel system can be estimated.

3.1. Simplified sloshing solution

It is assumed that the sloshing potential φ is given, instead of Eq. (16), by the following approximate expression:

$$\varphi(r, \theta, t) = \dot{q}_1 r \sin\theta + \dot{q}_2 r^2 \sin 2\theta. \tag{38}$$

Applying the boundary conditions, the system of ordinary differential equations (21) reduces to only one equation:

$$\ddot{q}_1 + \left(\frac{3\pi g}{8R}\right) q_1 = -\ddot{X}, \tag{39}$$

which is a linear oscillator equation. The frequency of the oscillator $\omega_S = \sqrt{3\pi g/8R}$ offers an approximation of the first sloshing frequency of the system. Once $q_1(t)$ is computed from Eq. (39), function $q_2(t)$ is calculated by

$$q_2 = -\left(\frac{3\pi}{16R}\right) q_1. \tag{40}$$

Dissipation effects may be introduced through a damping term in Eq. (39),

$$\ddot{q}_1 + 2\omega_S \xi_S \dot{q}_1 + \omega_S^2 q_1 = -\ddot{X}, \tag{41}$$

where ξ_S is the damping ratio.

Conducting an appropriate integration as indicated by Eq. (33), a simple expression is obtained for the sloshing force:

$$f_S = -(m_L/2)\ddot{q}_1 = -m_S \ddot{q}_1, \tag{42}$$

where m_S is half the liquid mass per unit length m_L, and it is referred to as ''sloshing'' mass. The total horizontal force f_T is the sum of the force associated with sloshing f_S, the force due to uniform motion of the liquid f_U, and the inertia force of the container f_C so that

$$f_T = -m_T \ddot{X} - m_S \ddot{q}_1, \tag{43}$$

where

$$m_T = m_L + m_C \tag{44}$$

is the total moving mass of the liquid–vessel system. Furthermore, using the following change of variable:

$$u = q_1 + X, \tag{45}$$

the total force expression in Eq. (43) becomes

$$f_T = -m_I \ddot{X} - m_S \ddot{u}, \tag{46}$$

where

$$m_I = m_T - m_S. \tag{47}$$

Eqs. (41), (45) and (46) motivate the consideration of an equivalent mechanical model shown in Fig. 2, for the response of a semi-circular disk under transverse excitation. The model is similar to mechanical models proposed elsewhere for rectangular and vertical cylindrical tanks (Abramson, 1966). Function $u(t)$ corresponds to the so-called ''convective'' motion and the corresponding mass m_S is the ''convective'' or ''sloshing'' mass. The other mass m_I represents the so-called ''impulsive'' mass and expresses the mass accelerating with the external source.

3.2. Application: simplified fluid–vessel interaction

Based on the simplified sloshing solution presented in the previous section, and considering a beam-type deformation of the cylindrical vessel, it is possible to develop an approximate formulation for the coupled response of the fluid–vessel system.

Motivated by practical applications, the cylindrical vessel is considered rather long (with length-to-radius ratio $L/R \geqslant 10$) and thick (with radius-to-thickness ratio $R/h \leqslant 100$), so that it exhibits a beam-type deformation while its cross-section remains circular (undeformed). Thus, the motion of the cylindrical container is directly determined by the motion of the cylinder axis, which is decomposed in two parts (Fig. 3), the motion of the supports $X_g(t)$, independent of z coordinate, and the motion due to the deformation of the container described by a function $y(z,t)$:

$$X(z,t) = X_g(t) + y(z,t). \tag{48}$$

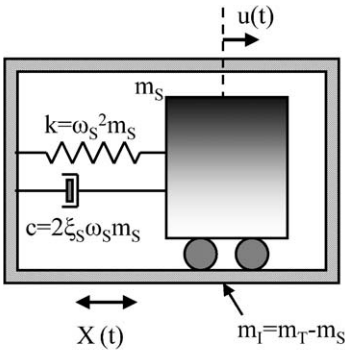

Fig. 2. Equivalent mechanical model for two dimensional analysis.

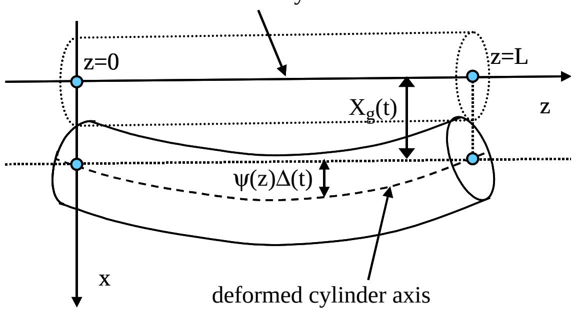

Fig. 3. Schematic representation of beam-type deformation of horizontal cylinder, simply supported at $z = 0$ and L.

An admissible function $\psi(z)$ is considered for the container deformation:

$$y(z,t) = \psi(z)\Delta(t), \tag{49}$$

and

$$X(z,t) = X_g(t) + \psi(z)\Delta(t), \tag{50}$$

so that the deformable vessel becomes a generalized single-degree-of-freedom system. In such a case the vessel undergoes a nonuniform motion with respect to the z coordinate and the sloshing solution is three dimensional. Nevertheless, in the majority of practical applications, the sloshing frequency ω_S is significantly smaller than the fundamental frequency of the container and, therefore, it is assumed that the two dimensional sloshing solution for the rigid container, as expressed by Eq. (41), is still valid for every cross-section. Consequently, for the cross-section corresponding to coordinate z, the following equation is considered:

$$\frac{\partial^2 q_1}{\partial t^2} + 2\xi_S\omega_S\frac{\partial q_1}{\partial t} + \omega_S^2 q_1 = -\frac{\partial^2 X}{\partial t^2}, \tag{51}$$

where $q_1 = q_1(z,t)$. From Eqs. (48)–(50), the unknown sloshing function $q_1(z,t)$ is the sum of two parts, one corresponding to the ground motion $q_g(t)$, and the other $q(t)$ corresponding to the tank motion relative to the ground motion,

$$q_1(z,t) = q_g(t) + \psi(z)q(t); \tag{52}$$

so that the unknown functions $q_g(t)$ and $q(t)$ satisfy the following equations:

$$\ddot{q}_g + 2\xi_S\omega_S\dot{q}_g + \omega_S^2 q_g = -\ddot{X}_g, \tag{53}$$

$$\ddot{q} + 2\xi_S\omega_S\dot{q} + \omega_S^2 q = -\ddot{\Delta}. \tag{54}$$

In the present formulation, $q(t)$ expresses the effects of wall deformation on sloshing, as indicated in Eq. (54).

Furthermore, from Eq. (43), the total lateral force per unit length of the cylinder at cross-section z is

$$f_T(z,t) = -m_S\frac{\partial^2 q_1}{\partial t^2} - m_T\frac{\partial^2 X}{\partial t^2} = -m_S\ddot{q}_g - m_T\ddot{X}_g - m_S\psi(z)\ddot{q} - m_T\psi(z)\ddot{\Delta}. \tag{55}$$

Equilibrium of the beam requires that

$$EI\left(\frac{\partial^4 y}{\partial z^4}\right) = f_T(z,t), \tag{56}$$

where EI is the bending stiffness of the beam-like cylinder. Using an arbitrary admissible function $w(z)$ and assuming that the cylinder is simply supported at the two ends ($z = 0$ and $z = L$), the weak form of the above equilibrium equation is obtained

$$\int_0^L EI\ y''(z,t)w''(z)\,\mathrm{d}z = \int_0^L f_T w(z)\,\mathrm{d}z, \tag{57}$$

where $(\ )''$ denotes double differentiation with respect to z. In the context of a Galerkin-type solution procedure, the trial function is approximated as follows:

$$w(z) = A_w\psi(z), \tag{58}$$

where A_w is an arbitrary number. Using Eqs. (55) and (57), the following dynamic equilibrium equation is obtained:

$$K_b \Delta = \int_0^L f_T \psi(z)\,\mathrm{d}z = -M_S \ddot{q}_g - M_T \ddot{X}_g - M'_S \ddot{q} - M'_T \ddot{\Delta}, \tag{59}$$

where the "generalized masses" M_S, M_T, M'_S and M'_T, and the "generalized bending stiffness" K_b are given by the following expressions:

$$M_S = \int_0^L m_S \psi(z)\,\mathrm{d}z, \quad M_T = \int_0^L m_T \psi(z)\,\mathrm{d}z, \quad M'_S = \int_0^L m_S \psi^2(z)\,\mathrm{d}z,$$
$$M'_T = \int_0^L m_T \psi^2(z)\,\mathrm{d}z, \quad K_b = \int_0^L EI \psi''^2(z)\,\mathrm{d}z. \tag{60}$$

In Eqs. (60), EI and m_T are constant along the cylinder, because of constant cylinder thickness. Furthermore, EI and m_T can be evaluated as follows:

$$EI \simeq E\pi \left(R + \frac{h}{2}\right)^3 h, \tag{61}$$

$$m_T \simeq m_L + 2\rho_C \pi (R+h) h, \tag{62}$$

where E and ρ_C are the elasticity modulus and the mass density of the container material, and h is the thickness of the container wall. Structural damping can be introduced through a term proportional to $\dot{\Delta}$, so that

$$M_S \ddot{q}_g + M_T \ddot{X}_g + M'_S \ddot{q} + M'_T \ddot{\Delta} + C_b \dot{\Delta} + K_b \Delta = 0, \tag{63}$$

where C_b is a damping coefficient. The three Eqs. (53), (54) and (63) may be written in the following matrix form:

$$[\mathscr{M}]Q + [\mathscr{C}]Q + [\mathscr{K}]Q = -\{\mathscr{T}\}\ddot{X}_g, \tag{64}$$

where

$$[\mathscr{M}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ M_S & M'_S & M'_T \end{bmatrix}, \quad [\mathscr{C}] = \begin{bmatrix} 2\xi_S\omega_S & 0 & 0 \\ 0 & 2\xi_S\omega_S & 0 \\ 0 & 0 & C_b \end{bmatrix},$$
$$[\mathscr{K}] = \begin{bmatrix} \omega_S^2 & 0 & 0 \\ 0 & \omega_S^2 & 0 \\ 0 & 0 & K_b \end{bmatrix}, \quad \{\mathscr{T}\} = \begin{bmatrix} 1 \\ 0 \\ M_T \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} q_g \\ q \\ \Delta \end{bmatrix}, \tag{65}$$

which can readily be solved for the unknown functions, $q_g(t)$ $q(t)$ and $\Delta(t)$. Subsequently, the total horizontal force on the vessel wall is computed as follows:

$$F_T = \int_0^L f_T(z,t)\,\mathrm{d}z = -(m_S L)\ddot{q}_g - (m_T L)\ddot{X}_g - M_S \ddot{q} - M_T \ddot{\Delta}. \tag{66}$$

Note that in the case of a nondeformable container, the third and the fourth term on the right-hand side vanish, and the above equation reduces to Eq. (43).

The three natural frequencies $\omega_{(i)}$ of the undamped coupled fluid–vessel system ($\xi_S = \xi_b = 0$) are the roots of the following equation:

$$\det([\mathscr{K}] - \omega^2[\mathscr{M}]) = 0, \tag{67}$$

and their analytic expressions are

$$\omega_{(1)}^2 = \omega_S^2, \tag{68}$$

$$\omega_{(2),(3)}^2 = \frac{K_b}{2(M'_T - M'_S)} \left[\left(1 + \frac{\omega_S^2}{\omega_b^2}\right) \pm \sqrt{\left(1 - \frac{\omega_S^2}{\omega_b^2}\right)^2 + 4\left(\frac{\omega_S^2}{\omega_b^2}\right)\frac{M'_s}{M'_T}} \right], \tag{69}$$

where $\omega_b^2 = K_b/M'_T$. In practical applications ($L/R \geqslant 10$, $R/h \leqslant 100$), it is readily shown that $\omega_S^2 \ll \omega_b^2$, so that the two natural frequencies in Eq. (69) become

$$\omega_{(2)}^2 \simeq \omega_S^2 \tag{70}$$

$$\omega_{(3)}^2 \simeq \omega_I^2 = \frac{K_b}{M_T' - M_S'} = \frac{K_b}{M_I'}. \tag{71}$$

The denominator $M_I' = M_T' - M_S'$ in Eq. (71) expresses the difference between the generalized total mass M_T' and the generalized sloshing mass M_S' and can be regarded as a "generalized impulsive mass", analogous to m_I defined in Eq. (47). In such a case, the dynamic response of the coupled system is governed by two natural frequencies, namely the sloshing frequency ω_S, which represents the motion of the liquid with respect to the vessel, and the "impulsive" frequency ω_I, given by Eq. (71), which represents the motion of the mass accelerating with the external source. Following the definition of ω_I, which expresses the motion of the generalized impulsive mass M_I', the damping coefficient can be computed as follows:

$$C_b = 2\xi_b \omega_I M_I', \tag{72}$$

where ξ_b is the structural damping ratio.

Finally, it is noted that this simplified coupled formulation as expressed by Eqs. (64) and (66) can be employed for approximating the coupled response of the fluid–vessel system for an arbitrary liquid depth of the horizontal cylinder, provided that appropriate values for the sloshing mass m_S and the sloshing frequency ω_S are employed. Calculation of such values requires a numerical solution of the corresponding sloshing problem, and it is out of the scope of the present study.

4. Response under harmonic excitation

The sloshing solution is significantly simplified and semi-analytical results are obtained when the container undergoes a harmonic motion

$$\dot{X}(t) = U\mathrm{e}^{-\mathrm{i}\omega t}, \tag{73}$$

where U is the velocity amplitude, and ω is the angular frequency of the external excitation source. Assuming steady state conditions, functions $\dot{q}_n(t)$ in Eq. (15) become

$$\dot{q}_n(t) = a_n \mathrm{e}^{-\mathrm{i}\omega t}, \tag{74}$$

and the following infinite system of linear algebraic equations is obtained for an nondeformable (rigid) container:

$$(-\omega^2[\mathbf{M}] + [\mathbf{K}])\{a\} = -\omega^2 U\{\gamma\}, \tag{75}$$

which is analogous to Eqs. (21). In the above system, the square matrix [**M**], the diagonal matrix [**K**] and the vector $\{\gamma\}$ are given by Eqs. (22)–(24), whereas $\{a\}$ is the unknown vector with components a_{2n-1}, n = 1,2,3... . In the presence of damping, the following set of algebraic equations are obtained:

$$(-\omega^2[\mathbf{M}] - \mathrm{i}\omega[\mathbf{C}] + [\mathbf{K}])\{a\} = -\omega^2 U\{\gamma\}, \tag{76}$$

which is analogous to Eqs. (29). Note that when $U = 0$ the system of algebraic equations (76) is reduced to a homogeneous system (i.e. an eigenvalue problem), and its solution provides the sloshing frequencies and modes. Furthermore, when $U = 0$ and $[\mathbf{C}] = 0$ in Eqs. (76), the resulting eigenvalue problem is identical to the one obtained by Evans and Linton (1993).

In nondeformable containers, an estimate of the externally induced sloshing effects on the overall response can be obtained from the computation of the added mass coefficient C_a:

$$C_a = \mathscr{R}e\left[\frac{f_S}{f_U + f_C}\right], \tag{77}$$

where $\mathscr{R}e$[ ] denotes the real part of the $f_S/(f_U+f_C)$ ratio. Furthermore, a measure of the dissipation mechanism is offered by the dimensionless damping coefficient C_v:

$$C_v = \mathscr{I}m\left[\frac{f_S}{f_U + f_C}\right], \tag{78}$$

where $\mathscr{I}m$[ ] denotes the imaginary part of the $f_S/(f_U+f_C)$ ratio.

In particular, assuming steady state conditions, the simplified sloshing formulation expressed by Eq. (38) results in the following closed-form expressions for the sloshing potential φ, the sloshing force f_S and the C_a and the C_v

coefficients:

$$\varphi(r,\theta,t)=\frac{\lambda^2}{(1-\lambda^2)-2\mathrm{i}\lambda\xi_S}\left(1-\frac{3\pi r}{8R}\cos\theta\right)r\sin\theta\mathrm{e}^{-\mathrm{i}\omega t}, \tag{79}$$

$$f_S=(\mathrm{i}\omega)m_S U\frac{\omega^2}{(\omega_S^2-\omega^2)-2\mathrm{i}\xi_S\omega_S\omega}\mathrm{e}^{-\mathrm{i}\omega t}, \tag{80}$$

$$C_a=\frac{1}{2}\frac{\lambda^2(1-\lambda^2)}{(1-\lambda^2)^2+(2\lambda\xi_S)^2}, \tag{81}$$

$$C_v=\frac{1}{2}\frac{2\lambda^3\xi_S}{(1-\lambda^2)^2+(2\lambda\xi_S)^2}, \tag{82}$$

where $\lambda=\omega/\omega_S$.

In the case of deformable horizontal cylindrical containers under the formulation proposed in the previous section, steady state conditions expressed by Eqs. (73) and (74) result in analytical expressions for $q_g(t)$, $q(t)$ and $\Delta(t)$. More specifically, in the absence of damping ($\xi_S=\xi_b=0$),

$$\dot{q}_g(t)=\frac{\omega^2 U}{\omega_S^2-\omega^2}\mathrm{e}^{-\mathrm{i}\omega t}, \tag{83}$$

$$\dot{q}(t)=\frac{M_T\omega_S^2-M_I\omega^2}{\mathscr{D}}\omega^4 U\mathrm{e}^{-\mathrm{i}\omega t}, \tag{84}$$

$$\dot{\Delta}(t)=(\omega_S^2-\omega^2)\frac{M_T\omega_S^2-M_I\omega^2}{\mathscr{D}}\omega^2 U\mathrm{e}^{-\mathrm{i}\omega t}, \tag{85}$$

where

$$\mathscr{D}=\det([\mathscr{K}]-\omega^2[\mathscr{M}])=(\omega_S^2-\omega^2)[(\omega_S^2-\omega^2)(K_b-\omega^2 M_T')-\omega^4 M_S'], \tag{86}$$

and

$$M_I=M_T-M_S \tag{87}$$

is a "generalized impulsive mass", analogous to M_I'. The total horizontal force F_T is obtained from Eq. (66):

$$F_T=(\mathrm{i}\omega)m_T L U\mathrm{e}^{-\mathrm{i}\omega t}\left[1+\frac{m_S}{m_T}\frac{\omega^2}{\omega_S^2-\omega^2}+\frac{M_S}{m_T L}\omega^4\frac{M_T\omega_S^2-M_I\omega^2}{\mathscr{D}}+\frac{M_T}{m_T L}\frac{M_T\omega_S^2-M_I\omega^2}{\mathscr{D}}\omega^2(\omega_S^2-\omega^2)\right]. \tag{88}$$

Note that in the case of a nondeformable (rigid) vessel only the first two terms in the brackets are present, whereas the remaining two terms in the brackets express the influence of the fluid–vessel interaction.

5. Results

The solution of the eigenvalue sloshing problem is presented first. Subsequently, the response of nondeformable half–full cylindrical vessels under harmonic excitation and a seismic event is examined. Finally, the coupled response of the fluid–vessel system is obtained and the significant effects of container deformation are demonstrated.

5.1. Sloshing frequencies and modes

The eigenvalue problem is studied, considering the solution of Eqs. (76), with $U=0$ (no external excitation). Due to the nonorthogonality of the spatial functions the sloshing frequency values ω_j, $j=1,2,\ldots$, in the present formulation depend on the truncation size of the series expansion N. The convergence rate and the expected accuracy of the eigenvalues are demonstrated numerically, increasing the value of truncation size N ($n\leqslant N$ in Eq. 16). In Fig. 4 the variation of the first three normalized sloshing frequencies $\omega\sqrt{R/g}$ is presented in terms of the truncation size N for zero dissipation. From the numerical point of view, the results indicate that the convergence rate is quite rapid, and that faster convergence is obtained in lower sloshing frequencies. The required truncation size N to obtain accurate results up to three significant figures for the first, second and third normalized eigenvalue is $N=4$, $N=8$ and $N=12$, respectively. The values of the eigenfrequencies are in very good agreement with experimental results (Abramson, 1966),

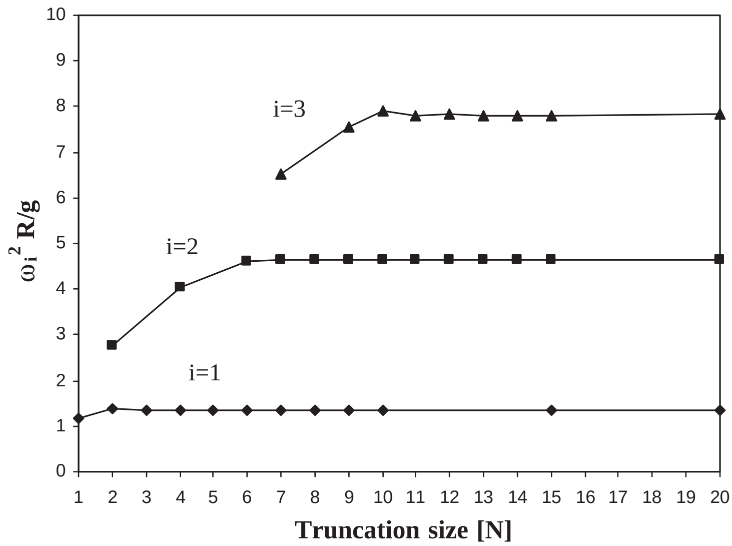

Fig. 4. Variation of the first three eigenfrequencies with respect to the truncation size N.

and other numerical or semi-analytical predictions (Budiansky, 1960; McIver and McIver, 1993). When damping is present the eigenvalues of the system become complex because of energy dissipation effects. The convergence rate of the real and imaginary parts of the complex eigenfrequencies of the damped system is similar to the corresponding eigenfrequencies of the undamped system. In Fig. 5, sloshing modes corresponding to the first three frequencies are depicted in terms of their free surface configuration. Furthermore, it is noted that the normalized value of $\omega_S(\omega_S\sqrt{R/g} = 1.085)$ based on the simplified methodology ($N=1$) offers a reasonable approximation of the first converged sloshing frequency $\omega_1(\omega_1\sqrt{R/g} = 1.164)$ and can be used in practical engineering applications.

5.2. Response of rigid vessels under harmonic excitation

The response of rigid (nondeformable) half–full horizontal cylinders under harmonic transverse excitation can be expressed in terms of the added mass coefficient C_a and the dimensionless damping coefficient C_v. The C_a and C_v values are plotted in Figs. 6 and 7, respectively, in terms of the normalized external excitation frequency (ω^2R/g). Damping is considered in the form of Eq. (30) with ∂_0 equal to 0, 0.34 and 0.68 and $\partial_1=0$. Fig. 6 shows that for the case of zero damping, the response is characterized by large increases in the C_a value in the vicinity of resonant frequencies. There is a sign reversal in C_a at each resonant frequency. When $C_a<0$, the sloshing force f_S is out-of-phase with the container displacement (i.e. the "uniform motion" force f_U), resulting in a reduction of the total force amplitude. The extreme values of C_a close to the resonant frequencies are significantly reduced when damping is present, and the resonant effect of the higher natural frequencies almost disappears. The large values of C_a for a wide range of excitation frequencies indicate the significant effects of hydrodynamic sloshing on the overall response. Fig. 7 presents the corresponding results for the dimensionless damping coefficient C_v. The C_v value exhibits a peak near the first resonant frequency, and much smaller peaks for the higher resonant frequencies. When the damping parameter value is increased, the peaks become smaller and smoother. The converged C_a values in the region of the dominant frequency ω_1 are compared in Fig. 8 with those obtained from the simplified formulation [Eq. (81)]. The comparison shows a reasonable agreement between the converged values of the horizontal cylinder and the values from the simplified methodology.

5.3. Transient response of rigid half–full horizontal cylinders

The efficiency of the proposed methodology to handle an arbitrary type of external excitation is demonstrated calculating the response of a half–full rigid horizontal cylindrical vessel, under an irregular input function $X(t)$. A half–full vessel is considered, with radius $R=1$ m and liquid density $\rho=1000\,\text{kg/m}^3$, subjected to the El Centro seismic ground motion (Fig. 9). Its length and its thickness are equal to $L=6$ m and $h=0.02$ m, respectively, so that it is practically nondeformable. Therefore, the problem is two dimensional and the response is obtained from the solution of

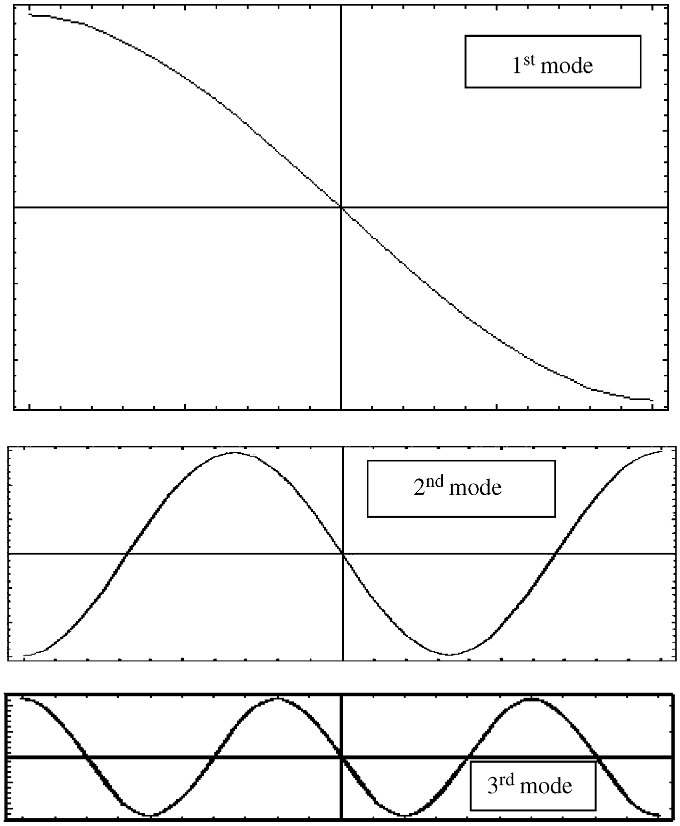

Fig. 5. Free surface elevation for the first three sloshing modes.

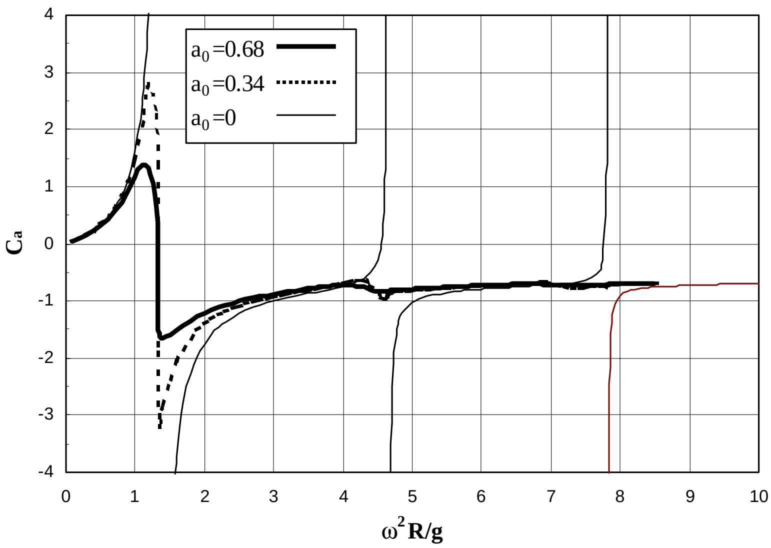

Fig. 6. Converged value of C_a in terms of external excitation frequency ω^2R/g ($\partial_1 = 0$).

the linear system of Eqs. (29) implementing a fourth-order Runge–Kutta scheme in Matlab programming, where the time step Δt is chosen equal to 0.020 s.

The dependence of the maximum value of the total force $(f_T L)_{\max}$ on the truncation size N is presented in Table 1, and shows that consideration of few terms of the series in the transverse direction (e.g. $n \leqslant N = 5$) is adequate to provide

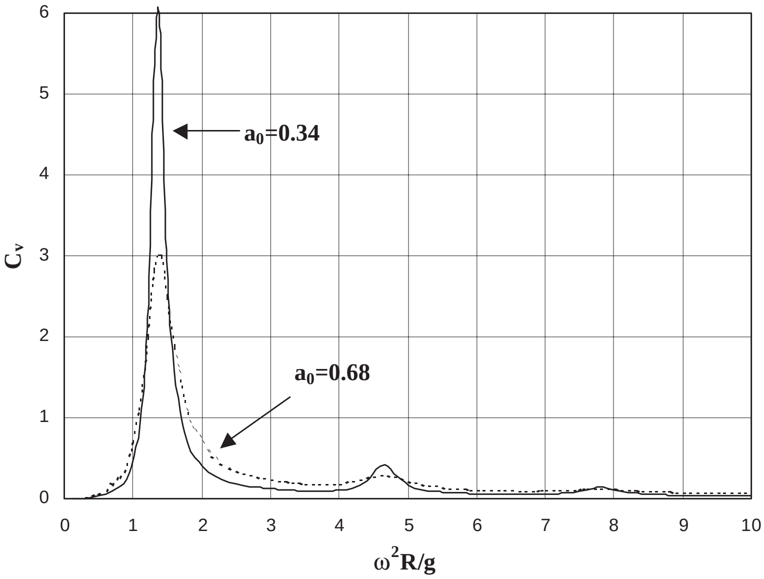

Fig. 7. Converged value of C_v in terms of external excitation frequency ω^2R/g ($\partial_1 = 0$).

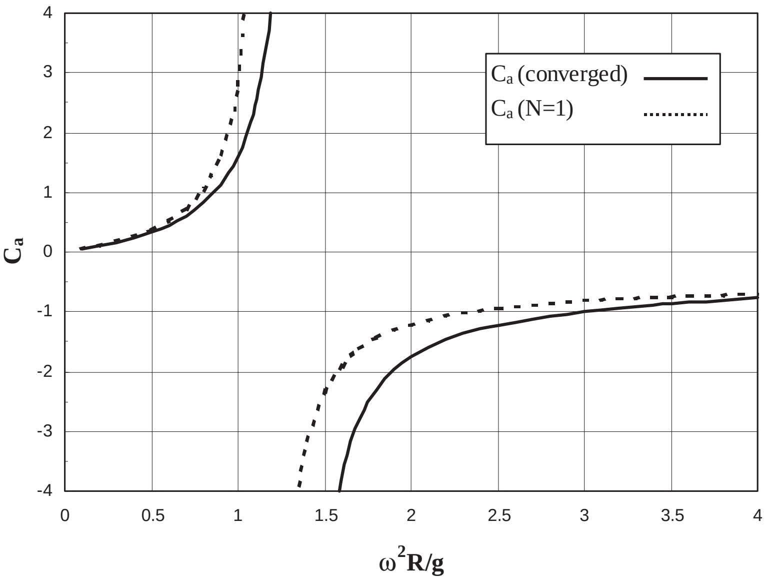

Fig. 8. Values of C_a in terms of external excitation frequency ω^2R/g around the first sloshing frequency ($\omega_1^2R/g = 1.355, \omega_S^2R/g = 1.178$).

quite accurate results for engineering purposes. In addition, the simplified formulation ($N = 1$) offers a fairly good approximation of the converged $(f_TL)_{\max}$ value.

Figs. 10(a), (b) and (c) show the uniform force f_UL, the force associated with sloshing f_SL and the total force f_TL, respectively for the cylindrical vessel under consideration, under the El Centro earthquake. The f_S and f_T values are obtained with a truncation size $N = 8$, and the maximum total force f_TL is 29.4 kN (at 4.14 s). The results show that sloshing force counteracts the uniform motion force and this is due to the fact that the dominant earthquake excitation frequencies are significantly larger than the dominant sloshing frequency ω_1, so that f_S is out-of-phase with f_U.

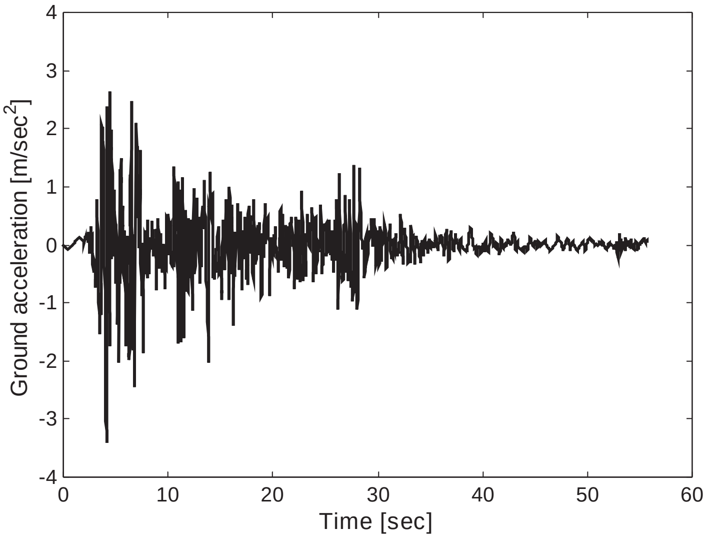

Fig. 9. El Centro ground motion (source: http://www.vibrationdata.com/elcentro.htm).

Table 1
Convergence of the maximum value of the total force with respect to truncation size N; $\alpha_0=0.34$, $\alpha_1=0$, R=1 m, L=6 m, h=0.02 m, ρ=1000 kg/m^3, $g=9.81$ m/s^2

N	$(f_T L)_{\max}$ in kN
1	33.3
2	31.1
3	30.1
4	29.3
5	29.4
6	29.4
7	29.4
8	29.4

5.4. Dynamic analysis and seismic response of fluid–vessel system

The dynamic response of horizontal cylindrical pressure vessels under seismic loading is of particular importance in petrochemical industry applications. Herein, a half–full VCM horizontal cylindrical container is examined. The radius, thickness and the length are $R=1$ m, $h=0.01$ m and $L=20$ m, respectively and the liquid has a density $\rho=850$ kg/m^3. The vessel is simply supported at $z=0$ and $z=L$, and a sinusoidal function $\psi(z)$ is employed:

$$\psi(z) = \sin\left(\frac{\pi z}{L}\right). \tag{89}$$

Therefore, $M_S = 0.637 m_S L, M_T = 0.637 m_T L, M'_S = 0.5 m_S L, M'_T = 0.5 m_T L, K_b = 48.7 EI/L^3$. The sloshing frequency of the liquid ω_S is 3.40 rad/s. Furthermore, the values of ω_I and ω_b are 59 and 47 rad/s, respectively, significantly larger than ω_S.

The vessel is subjected to the El Centro earthquake (Fig. 9). The response of the deformable vessel is shown in Fig. 11(a), in terms of the total force F_T, obtained from Eq. (66). The maximum value of F_T is 109.92 kN, at 4.26 s. Fig. 11(b) shows the response of the same half–full container considered as rigid (i.e. neglecting container wall deformation). In such a case, the total force F_T is calculated again from Eq. (66), neglecting the third and the fourth term of the right-hand side, and the maximum value of F_T is 71.26 kN at 13.83 s. The above values indicate clearly that the total force on the wall of the deformable container is significantly higher than the corresponding total force on the

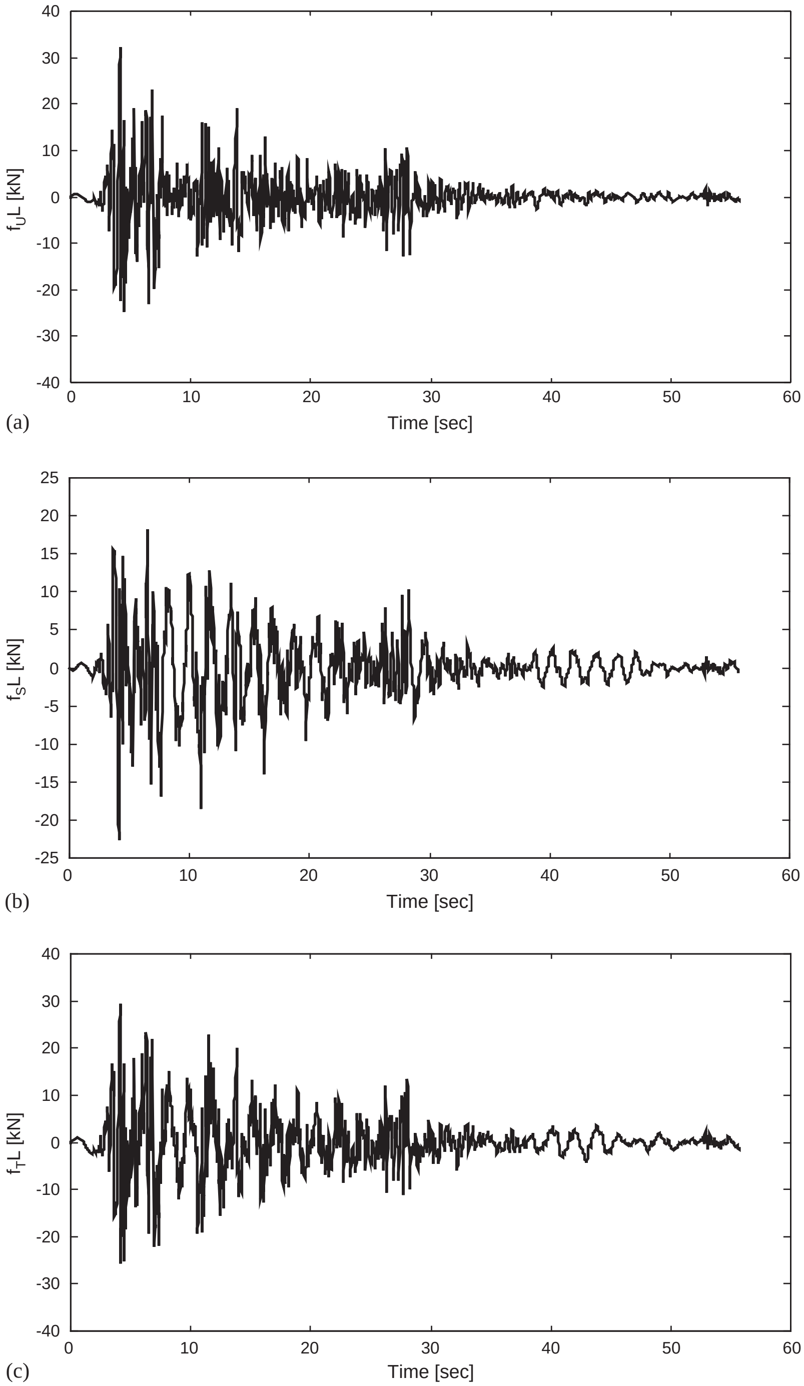

Fig. 10. Response of a half–full cylindrical vessel subjected to the El Centro earthquake in its transverse direction with damping ($\partial_0 = 0.34$, $\partial_1 = 0$). (a) uniform motion force $f_U L$, (b) force associated with sloshing $f_S L$ and (c) total force $f_T L$.

wall of an equivalent rigid container. This difference is attributed to the amplification of ground motion, due to the flexibility of the impulsive part of the system.

Figs. 12a and b depict the time variation of $q_g(t)$ and $q(t)$ for the vessel under consideration, subjected to the El Centro earthquake. The amplitude of $q_g(t)$ is significantly larger than the amplitude of $q(t)$. This demonstrates the small

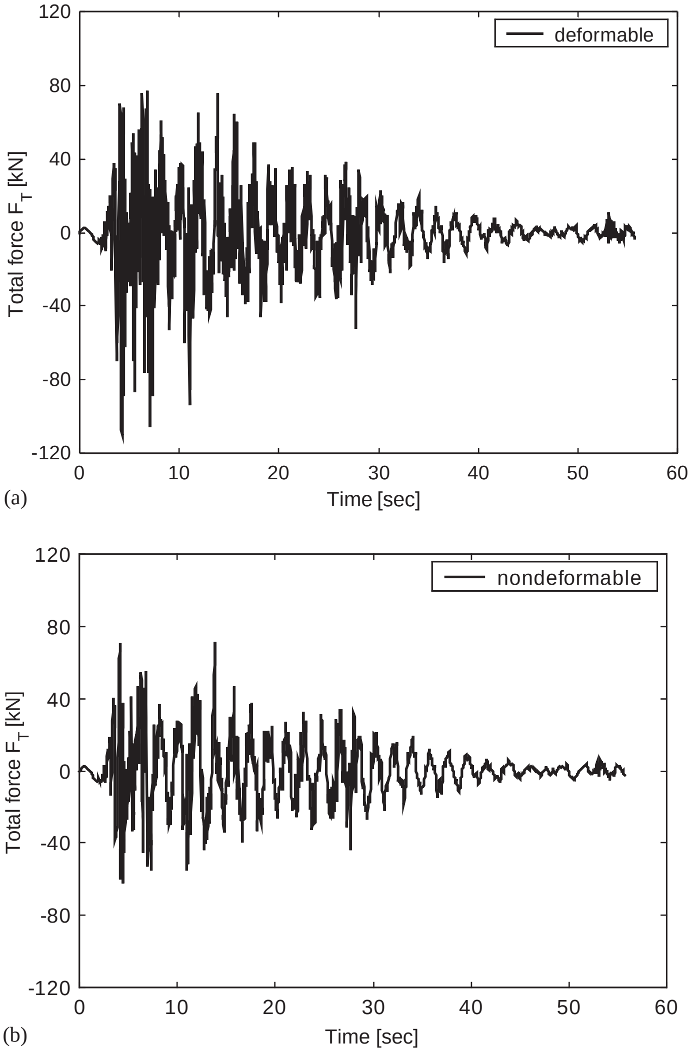

Fig. 11. Total forces F_T assuming (a) deformable container and (b) rigid container ($N=1$, $\xi_S=2\%$, $\xi_b=3\%$).

influence of wall deformation on sloshing in horizontal cylinders, which is consistent with previous observations in vertical cylindrical containers [e.g. Veletsos and Yang (1977); Fischer, (1979)].

6. Conclusions

A mathematical model is developed for externally induced liquid sloshing in half–full horizontal cylindrical containers, in the transverse direction. In this configuration, the problem formulation is not separable and the general solution of the sloshing potential is written as a series expansion of arbitrary time functions and its associated nonorthogonal spatial functions. The formulation allows for a semi-analytical solution, results in a system of linear ordinary differential equations, and enables the prediction of sloshing effects under any form of external excitation, in a simple and efficient manner. A relatively small truncation of the series is adequate to provide good results. In the case of harmonic excitation, the results are expressed in terms of the added force coefficient C_a and the dimensionless damping coefficient C_v.

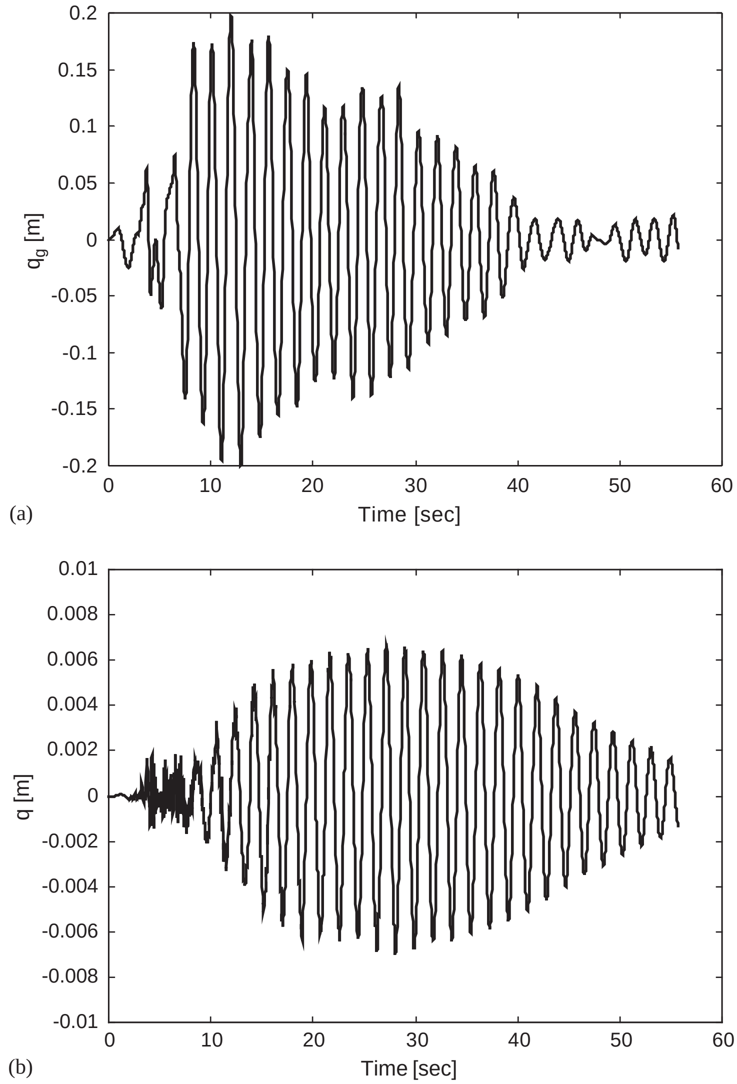

Fig. 12. Functions $q_g(t)$ and $q(t)$ assuming deformable container ($N=1$, $\xi_S=2\%$, $\xi_b=3\%$).

Considering only the first two terms of the general solution, a simplified solution of the problem is obtained, which results in closed-form expressions for the response under harmonic excitation. Despite its simplicity, this formulation provides results of fairly good accuracy in comparison with the converged solution.

Finally, the effects of vessel wall deformation are examined. Considering the aforementioned simplified sloshing solution and assuming a beam-type deformation of the horizontal cylinder, a coupled formulation is developed, which approximates the dynamic response of the fluid–vessel system. Using this formulation, the dynamic response of a deformable half–full vessel is obtained under a real seismic event. It is found that wall deformation may have a considerable effect on the total seismic force, whereas the influence of wall deformation on sloshing is rather insignificant.

Acknowledgements

The present study was partially supported by the Earthquake Planning & Protection Organization, Athens, Greece.

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